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Generalizations of the Camassa–Holm equation

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Abstract
We classify generalized Camassa–Holm-type equations which possess infinite hierarchies of higher symmetries. We show that the obtained equations can be treated as negative flows of integrable quasi-linear scalar evolution equations of orders 2, 3 and 5. We present the corresponding Lax representations or linearization transformations for these equations. Some of the obtained equations seem to be new.

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1. Introduction

In recent years, there has been a growing interest in integrable non-evolutionary partial differential equations of the form

$$\left(1-D_x^2\right)u_t = F(u, u_x, u_{xx}, u_{xxx}, \ldots), \qquad u = u(x,t), \qquad D_x = \frac{\partial}{\partial x}. \tag{1}$$

Here F is some function of u and its derivatives with respect to x. The most celebrated example of this type of equations is the Camassa–Holm equation [1]:

$$\left(1-D_x^2\right)u_t = 3uu_x - 2u_xu_{xx} - uu_{xxx}.$$

Another equivalent form of the Camassa–Holm equation is

$$m_t = 2mu_x + um_x, \qquad m = u - u_{xx}.$$

The Camassa–Holm equation is integrable by the inverse scattering transform. It possesses an infinite hierarchy of local conservation laws, bi-Hamiltonian structure and other remarkable properties of integrable equations. Despite its non-evolutionary form, the Camassa–Holm equation possesses an infinite hierarchy of local higher symmetries—indeed this equation can be viewed as an inverse flow of the equation $u_\tau = D_x(u-u_{xx})^{-\frac{1}{2}}$. Furthermore, the Camassa–Holm equation can be reduced via a reciprocal transformation to the first negative of the Korteweg–de Vries hierarchy (see also [2]). The Camassa–Holm equation possesses multi-phase peakon solutions (peaked soliton solutions with discontinuous derivatives at the peaks).

Until 2002, the Camassa–Holm equation was the only known integrable example of type (1), which possesses peakon solutions, when Degasperis and Procesi isolated another equation

$$\left(1 - D_x^2\right)u_t = 4uu_x - 3u_x u_{xx} - uu_{xxx},$$

or in a different form

$$m_t = 3mu_x + um_x, \qquad m = u - u_{xx},$$

which was also found to be integrable by the inverse scattering transform [5]. The Degasperis–Procesi equation also possesses infinitely many conservation laws, bi-Hamiltonian structure, etc. It also possesses an infinite hierarchy of local higher symmetries and can be seen as a non-local symmetry of a local evolutionary equation $u_\tau = \left(4 - D_x^2\right)D_x(u - u_{xx})^{-\frac{2}{3}}$. In fact, the Degasperis–Procesi equation can be reduced via a reciprocal transformation to the first negative flow of the Kaup–Kupershmidt hierarchy [3].

One may ask the following questions: are there other integrable equations of the form (1), and is it possible to classify all integrable equations of this type? The answer to both questions is positive.

The first classification result of equations of type (1) was obtained in [6] using the perturbative symmetry approach in the symbolic representation. In the symmetry approach the existence of infinite hierarchies of higher symmetries is adopted as a definition of integrability. The conditions of existence of higher symmetries are very restrictive and result in algorithmic and efficient integrability test. In particular, the following result was proved in [6].

Theorem 1. *If equation*

$$m_t = bmu_x + um_x, \qquad m = u - u_{xx}, \qquad b \in \mathbb{C}\backslash\{0\}$$

possesses an infinite hierarchy of (quasi-) local higher symmetries, then $b = 2, 3$.

Obviously, the case $b = 2$ corresponds to the Camassa–Holm equation, while $b = 3$ gives the Degasperis–Procesi equation.

In this article, we extend the classification result of [6] and apply the perturbative symmetry approach to isolate and classify more general class of integrable equations of the form (1). We assume that function F on the right-hand side is a homogeneous differential polynomial over $\mathbb{C}$, quadratic or cubic in u and its x-derivatives. The obtained list comprises 28 equations (see section 3) and some of these equations seem to be new to the best of our knowledge. The list includes an equation of the form

$$\left(1 - D_x^2\right)u_t = u^2 u_{xxx} + 3uu_x u_{xx} - 4u^2 u_x.$$

Integrability and multipeakon solutions of this equation have been recently studied in [7, 11]. For all the obtained equations we present their first non-trivial higher symmetries. We also give Lax representations or linearization transformations for most of the equations. We show that all the obtained equations can be treated as negative flows of integrable quasi-linear scalar evolution equations of orders 2, 3 or 5. The classification results of the latter ones can be found in [10].

2. Integrability test

In this section, we briefly recall the basic definitions and notations of the perturbative symmetry approach (for details see [6, 12]). We also present the integrability test [6], which we apply to isolate integrable generalizations of the Camassa–Holm equation.

2.1. Symmetries and approximate symmetries

In what follows, we shall consider the Camassa–Holm-type equation (1) with the right- hand side being a differential polynomial over $\mathbb{C}$.

Let $\mathcal{R}$ be a ring of differential polynomials in $u, u_x, u_{xx}, \ldots$ over $\mathbb{C}$. We shall adopt a notation

$$u_i \equiv D_x^i(u).$$

We shall often omit subscript 0 at u_0 and write u instead of u_0.

The ring $\mathcal{R}$ is a differential ring with a derivation

$$D_x = \sum_{i\geqslant 0} u_{i+1}\frac{\partial}{\partial u_i}.$$

The ring has a natural gradation with respect to degrees of nonlinearity in u and its x-derivatives:

$$\mathcal{R} = \bigoplus_{i\geqslant 0} \mathcal{R}_i,$$

$$\mathcal{R}_i = \{f(u, u_1, \ldots, u_k) \in \mathcal{R} | f(\lambda u, \lambda u_1, \ldots, \lambda u_k) = \lambda^i f(u, u_1, \ldots, u_k)\}, \qquad \lambda \in \mathbb{C}.$$

The space $\mathcal{R}_0 = \mathbb{C}$, $\mathcal{R}_1$ is a space of linear polynomials in $u, u_1, \ldots, \mathcal{R}_2$ is a space of quadratic polynomials, etc. It is convenient to introduce a notion of 'little oh' as

$$f = o(\mathcal{R}_p) \quad \Leftrightarrow \quad f \in \bigoplus_{i>p} \mathcal{R}_i.$$

Let us denote by $\mathcal{R}_+$ a differential ring without a unit:

$$\mathcal{R}_+ = \bigoplus_{i>0} \mathcal{R}_i.$$

Suppose that $F \in \mathcal{R}_+$ in equation (1). We can formally rewrite equation (1) as an evolutionary equation as

$$u_t = \Delta(F), \qquad \Delta = \left(1 - D_x^2\right)^{-1}. \tag{2}$$

Symmetries and conservation laws of this equation, if they exist, may also contain operator Δ in their structure and therefore we need an extension of the differential ring $\mathcal{R}_+$ with the operator Δ. The construction of such extension was first suggested in [9] for the evolutionary (2 + 1)-dimensional equations. For the Camassa–Holm-type equations it was first applied in [6]. For example, let us construct a sequence of spaces $\mathcal{R}_+^i$, $i = 0, 1, 2, \ldots,$ as follows:

$$\mathcal{R}_+^0 = \mathcal{R}_+, \qquad \mathcal{R}_+^1 = \overline{\mathcal{R}_+^0 \bigcup \Delta\left(\mathcal{R}_+^0\right)}, \qquad \mathcal{R}_+^{n+1} = \overline{\mathcal{R}_+^n \bigcup \Delta\left(\mathcal{R}_+^n\right)}.$$

The subscript n in $\mathcal{R}_+^n$ is the 'nesting depth' of the operator Δ. The extension construction is compatible with the natural gradation:

$$\mathcal{R}_+^n = \bigoplus_{i>0} \mathcal{R}_i^n, \qquad \mathcal{R}_i^n = \left\{ f[u] \in \mathcal{R}_+^n \,\middle|\, f[\lambda u] = \lambda^i f[u] \right\}, \qquad \lambda \in \mathbb{C}.$$

It is clear that $\Delta(F)$ in equation (2) belongs to $\mathcal{R}_+^1$. The symmetries of the equation may belong to $\mathcal{R}_+^k$ for some appropriate $k \geqslant 0$ and we introduce the following definition of a symmetry.

Definition 1. *A function $G \in \mathcal{R}_+^k$, $k \geqslant 0$, is called a generator of a symmetry of equation (2) if a differential equation*

$$u_\tau = G$$

is compatible with equation (2): $G_t - F_\tau = 0$.

We adopt the following definition of integrability.

Definition 2. *Equation (2) is integrable if it possesses an infinite hierarchy of symmetries.*

In addition to the definition of a symmetry we also introduce a definition of an approximate symmetry.

Definition 3. *A function $G \in \mathcal{R}_+^k$, $k \geqslant 0$, is called a generator of an approximate symmetry of degree p of equation (2) if $G_t - F_\tau = o\big(\mathcal{R}_p^k\big)$.*

Any equation

$$u_t = \Delta(F) = \Delta(F_1) + \Delta(F_2) + \cdots + \Delta(F_k), \qquad F_k \in \mathcal{R}_k,$$

possesses an infinite hierarchy of approximate symmetries of degree 1—these are symmetries of its linear part $u_t = \Delta(F_1)$. The condition of existence of approximate symmetries of degree 2 imposes strong restrictions on the equation. However, an equation may possess infinitely many of approximate symmetries of degree 2, but fail to possess approximate symmetries of degree 3. On the other hand, an integrable equation possesses infinitely many approximate symmetries of any degree. The degree of approximate symmetry can be viewed as a measure of the integrability. In many cases, the existence of approximate symmetries of sufficiently large degree implies integrability.

In order to derive the conditions of existence of symmetries and approximate symmetries, it is convenient to introduce the symbolic representation of the ring $\mathcal{R}_+$ and its extension.

2.2. Symbolic representation

We start by introducing the symbolic representation $\hat{\mathcal{R}}_+$ of $\mathcal{R}_+$. We first introduce the symbolic representation of spaces $\mathcal{R}_k$, $k = 1, 2, \ldots$.

(1) To a linear monomial $u_i \in \mathcal{R}_1$ we put into correspondence a symbol

$$u_i \longrightarrow \hat{u}\xi_1^i.$$

(2) To a quadratic monomial $u_i u_j \in \mathcal{R}_2$ we put into correspondence a symbol

$$u_i u_j \longrightarrow \frac{\hat{u}^2}{2}\big(\xi_1^i\xi_2^j + \xi_1^j\xi_2^i\big).$$

(3) We represent a generic $u_0^{n_0}u_1^{n_1}\cdots u_k^{n_k} \in \mathcal{R}_n$, $n = n_0 + n_1 + \cdots + n_k$ by a symbol

$$u_0^{n_0}u_1^{n_1}\cdots u_k^{n_k} \longrightarrow \hat{u}^n\big\langle \xi_1^0\cdots\xi_{n_0}^0\xi_{n_0+1}^1\cdots\xi_{n_0+n_1}^1\cdots\xi_n^k\big\rangle,$$

where the brackets $\langle * \rangle$ denote a symmetrization operation:

$$\langle f(\xi_1,\ldots,\xi_n)\rangle = \frac{1}{n!}\sum_{\sigma\in S_n} f(\xi_{\sigma(1)},\ldots,\xi_{\sigma(n)}).$$

We define addition, multiplication and derivation as follows. Let $f \in \mathcal{R}_i$ and $g \in \mathcal{R}_j$ be two monomials and their symbolic representation is given by $f \to \hat{u}^i a(\xi_1,\ldots,\xi_i)$ and $g \to \hat{u}^j b(\xi_1,\ldots,\xi_j)$. Then

$$f + g \longrightarrow \hat{u}^i a(\xi_1,\ldots,\xi_i) + \hat{u}^j b(\xi_1,\ldots,\xi_j)$$

and

$$f\cdot g \longrightarrow \hat{u}^{i+j}\langle a(\xi_1,\ldots,\xi_i)b(\xi_{i+1},\ldots,\xi_{i+j})\rangle.$$

In particular, if $i = j$, then $f + g \to \hat{u}^i\,(a(\xi_1,\ldots,\xi_i) + b(\xi_1,\ldots,\xi_i))$.

To a derivative of $f \to \hat{u}^i a(\xi_1, \dots, \xi_i)$, we put into correspondence

$$D_x(f) \longrightarrow \hat{u}^i a(\xi_1, \dots, \xi_i)(\xi_1 + \cdots + \xi_i).$$

This concludes the construction of the symbolic representation $\hat{\mathcal{R}}_+$ of the differential ring $\mathcal{R}_+$.

We also introduce a notion of a pseudo-differential formal series in the symbolic representation. We reserve a special symbol η for the operator D_x in the symbolic representation with an action rule

$$\eta(\hat{u}^n a(\xi_1, \dots, \xi_n)) = \hat{u}^n a(\xi_1, \dots, \xi_n)(\xi_1 + \cdots + \xi_n).$$

Let $f D_x^p$ and $g D_x^q$, $p, q \in \mathbb{Z}$, be two (pseudo)-differential operators and suppose that $f \to \hat{u}^i a(\xi_1, \dots, \xi_i)$ and $g \to \hat{u}^j b(\xi_1, \dots, \xi_j)$. Then for the symbolic representation of these operators we have

$$f D_x^p \longrightarrow \hat{u}^i a(\xi_1, \dots, \xi_i)\eta^p, \qquad g D_x^q \longrightarrow \hat{u}^j b(\xi_1, \dots, \xi_i)\eta^q.$$

For the addition and composition of pseudo-differential operators in the symbolic representation we have

$$f D_x^p + g D_x^q \longrightarrow \hat{u}^i a(\xi_1, \dots, \xi_i)\eta^p + \hat{u}^j b(\xi_1, \dots, \xi_i)\eta^q,$$
$$f D_x^p \circ g D_x^q \longrightarrow \hat{u}^{i+j} \langle a(\xi_1, \dots, \xi_i)(\eta + \xi_{i+1} + \cdots + \xi_{i+j})^p b(\xi_{i+1}, \dots, \xi_{i+j})\eta^q \rangle.$$

More generally we shall consider formal series in the form

$$A = a_0(\eta) + \hat{u} a_1(\xi_1, \eta) + \hat{u}^2 a_2(\xi_1, \xi_2, \eta) + \hat{u}^3 a_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots, \tag{3}$$

where functions $a_k(\xi_1, \dots, \xi_k, \eta)$ are symmetric functions with respect to arguments $\xi_1, \dots, \xi_k$. The addition rule of such series is obvious, while for composition of two monomials we have

$$\hat{u}^i a(\xi_1, \dots, \xi_i, \eta) \circ \hat{u}^j b(\xi_1, \dots, \xi_j, \eta)$$
$$= \hat{u}^{i+j} \langle a(\xi_1, \dots, \xi_i, \eta + \xi_{i+1} + \cdots + \xi_{i+j}) b(\xi_{i+1}, \dots, \xi_{i+j}, \eta) \rangle,$$

where the symmetrization operation is taken with respect to all arguments $\xi_1, \dots, \xi_{i+j}$, but not η.

We introduce a notion of *locality* of a pseudo-differential operator.

Definition 4. *Function $a(\xi_1, \dots, \xi_i, \eta)$ is called local if all coefficients $a_j(\xi_1, \dots, \xi_i)$ of its expansion in η at $\eta \to \infty$*

$$a(\xi_1, \dots, \xi_i, \eta) = \sum_{j<s} a_j(\xi_1, \dots, \xi_i)\eta^j$$

are symmetric polynomials in variables $\xi_1, \dots, \xi_i$. Formal series (3) is called local if all functions $a_j(\xi_1, \dots, \xi_j)$, $j = 1, 2, \dots$, in (3) are local.

To construct the symbolic representation of the extension of the ring $\mathcal{R}_+$ with the operator $\Delta = \left(1 - D_x^2\right)^{-1}$ it is enough to note that the symbolic representation of the operator Δ is

$$\Delta \longrightarrow (1 - \eta^2)^{-1}.$$

Indeed, if $f \in \mathcal{R}_k$ and $f \to \hat{u}^k a(\xi_1, \dots, \xi_k)$, then

$$\Delta(f) \longrightarrow \hat{u}^k \frac{a(\xi_1, \dots, \xi_k)}{1 - (\xi_1 + \cdots + \xi_k)^2}.$$

Using if necessary the addition and multiplication operations we thus can obtain the symbolic representation of any space $\mathcal{R}_+^j$.

In addition to the notion of locality of a pseudo-differential series we also introduce a notion of *quasi-locality*.

Definition 5. *A pseudo-differential operator*

$$\hat{u}^n a(\xi_1, \ldots, \xi_n, \eta) = \sum_{i<s} \hat{u}^n a_i(\xi_1, \ldots, \xi_n)\eta^i$$

is called quasi-local if for all $i < s$, $\hat{u}^n a_i(\xi_1, \ldots, \xi_n)$ *are symbolic representations of some elements from* $\mathcal{R}_n^k$ *for some* $k \geqslant 0$. *A formal series (3) is called quasi-local if all its terms are quasi-local.*

Finally, we introduce a notion of a Frechet derivative in the symbolic representation: let $f \in \mathcal{R}_k^n$, $k > 0$, $n \geqslant 0$, and its symbolic representation is given by $f \to \hat{f} = \hat{u}^k a(\xi_1, \ldots, \xi_k)$. Then the Frechet derivative f_* corresponds to

$$f_* \to \hat{f}_* = k\hat{u}^{k-1} a(\xi_1, \ldots, \xi_{k-1}, \eta).$$

2.3. Symmetries and approximate symmetries in the symbolic representation

Now we derive conditions of existence of symmetries and approximate symmetries of equation (2). We shall suppose that $F \in \mathcal{R}_+$ and thus we can rewrite equation (2) as

$$u_t = \Delta(F) = \Delta(F_1) + \Delta(F_2) + \cdots + \Delta(F_k), \qquad F_i \in \mathcal{R}_i, \qquad i = 1, 2, \ldots. \tag{4}$$

We write the symbolic representation of $\Delta(F)$ as

$$\Delta(F) \longrightarrow \hat{F} = \hat{u}\omega(\xi_1) + \hat{u}^2 a_1(\xi_1, \xi_2) + \cdots + \hat{u}^k a_{k-1}(\xi_1, \ldots, \xi_k). \tag{5}$$

By construction $a_i(\xi_1, \ldots, \xi_{i+1})$, $i = 1, \ldots, k-1$, are symmetric rational functions in $\xi_1, \ldots, \xi_{i+1}$ of the form

$$a_i(\xi_1, \ldots, \xi_{i+1}) = \frac{b_i(\xi_1, \ldots, \xi_{i+1})}{1 - (\xi_1 + \cdots + \xi_{i+1})^2},$$

where symmetric polynomials $b_i(\xi_1, \ldots, \xi_{i+1})$ are symbolic representations of differential polynomials F_{i+1}, $i = 1, 2, \ldots, k-1$. Similarly $\omega(\xi_1) = \tilde{\omega}(\xi_1)\big/\big(1 - \xi_1^2\big)$ and $\tilde{\omega}(\xi_1)$ is a symbolic representation of F_1. We shall suppose that F_1 is such that $\omega(\xi_1) \neq \text{const}\, \xi_1$.

Let $G \in \mathcal{R}_+^n$, $n \geqslant 0$, be a symmetry of (4). Without loss of generality we can suppose that

$$G = G_1 + G_2 + \cdots + G_m, \qquad G_i \in \mathcal{R}_i^n, \quad i = 1, \ldots, m.$$

Let

$$G \longrightarrow \hat{u}\Omega(\xi_1) + \hat{u}^2 A_1(\xi_1, \xi_2) + \cdots + \hat{u}^m A_{m-1}(\xi_1, \ldots, \xi_m) \tag{6}$$

be a symbolic representation of G, i.e. $\hat{u}^i A_{i-1}(\xi_1, \ldots, \xi_i)$, $i = 1, \ldots, m-1$, are symbolic representations of $G_i \in \mathcal{R}_i^n$ and thus are symmetric rational functions in $\xi_1, \ldots, \xi_i$.

The following proposition holds.

Proposition 1. *The function* $G \in \mathcal{R}_+^n$, $n \geqslant 0$, *with the symbolic representation (6) is a generator of a symmetry of equation (4) with the symbolic representation (5) if and only if*

$$A_1(\xi_1, \xi_2) = \frac{\Omega(\xi_1 + \xi_2) - \Omega(\xi_1) - \Omega(\xi_2)}{\omega(\xi_1 + \xi_2) - \omega(\xi_1) - \omega(\xi_2)} a_1(\xi_1, \xi_2),$$

$$
\begin{aligned}
A_m(\xi_1,\ldots,\xi_{m+1}) &= \frac{G^{\Omega}(\xi_1,\ldots,\xi_{m+1})}{G^{\omega}(\xi_1,\ldots,\xi_{m+1})} a_m(\xi_1,\ldots,\xi_{m+1}) \\
&+ G^{\omega}(\xi_1,\ldots,\xi_{m+1})^{-1} \cdot \left[\left\langle \sum_{j=1}^{m-1} \frac{m+1}{m-j+1} A_j\left(\xi_1,\ldots,\xi_j,\sum_{k=j+1}^{m+1}\xi_k\right) a_{m-j}(\xi_{j+1},\ldots,\xi_{m+1}) \right.\right. \\
&\left.\left. - \sum_{j=1}^{m-1} \frac{m+1}{j+1} a_{m-j}\left(\xi_1,\ldots,\xi_{m-j},\sum_{k=m-j+1}^{m+1}\xi_k\right) \cdot A_j(\xi_{m-j+1},\ldots,\xi_{m+1}) \right\rangle \right],
\end{aligned}
$$

where

$$
\begin{aligned}
G^{\omega}(\xi_1,\ldots,\xi_m) &= \omega\left(\sum_{n=1}^{m}\xi_n\right) - \sum_{n=1}^{m}\omega(\xi_n), \\
G^{\Omega}(\xi_1,\ldots,\xi_m) &= \Omega\left(\sum_{n=1}^{m}\xi_n\right) - \sum_{n=1}^{m}\Omega(\xi_n)
\end{aligned}
$$

and $\hat{u}^i A_{i-1}(\xi_1,\ldots,\xi_{i-1})$ *are symbolic representations of elements of* $\mathcal{R}_i^n$.

The proof follows from the compatibility conditions of equation (4) and $u_\tau = G$ (for details see [6]). Proposition 1 gives necessary and sufficient conditions of existence of an approximate symmetry of degree p. Indeed, if for a given equation (4) with the symbolic representation (5) $\hat{u}^i A_{i-1}(\xi_1,\ldots,\xi_i)$ are symbolic representations of elements of $\mathcal{R}_i^n$ for all $i = 1, 2, \ldots, p$, then G is an approximate symmetry of degree p. Note that if G is a symmetry, then it is completely determined by its linear part G_1. From proposition 1 it follows that to characterize a hierarchy of symmetries it is sufficient to characterize a hierarchy of admissible linear terms.

However, it is possible to derive the necessary conditions of existence of an infinite hierarchy of (approximate) symmetries without knowing the structure of admissible linear terms of the symmetries. To do so we introduce a notion of a formal recursion operator.

Definition 6. *A quasi-local formal series*

$$\Lambda = \phi(\eta) + \hat{u}\phi_1(\xi_1,\eta) + \hat{u}^2\phi_2(\xi_1,\xi_2,\eta) + \hat{u}^3\phi_3(\xi_1,\xi_2,\xi_3,\eta) + \cdots \tag{7}$$

is called a formal recursion operator for equation (4) if it satisfies

$$\Lambda_t = \hat{F}_* \circ \Lambda - \Lambda \circ \hat{F}_*, \tag{8}$$

where $\hat{F}_*$ *is a symbolic representation of a Frechet derivative of* F.

The following statement holds.

Theorem 2. *If equation (4) possesses an infinite hierarchy of higher symmetries, then it possesses a formal recursion operator (7) with* $\phi(\eta) = \eta$.

The proof of the theorem can be found in [6].

The equation $\Lambda_t = \hat{F}_* \circ \Lambda - \Lambda \circ \hat{F}_*$ can be resolved in terms of functions $\phi_i(\xi_1,\ldots,\xi_i,\eta)$.

Proposition 2. *Let* $\phi(\eta)$ *be an arbitrary function and formal series*

$$\Lambda = \phi(\eta) + \hat{u}\phi_1(\xi_1,\eta) + \hat{u}^2\phi_2(\xi_1,\xi_2,\eta) + \hat{u}^3\phi_3(\xi_1,\xi_2,\xi_3,\eta) + \cdots$$

be a solution of equation (8); then its coefficients $\phi_m(\xi_1, \ldots, \xi_m, \eta)$ *can be found recursively:*

$$\phi_1(\xi_1, \eta) = \frac{2(\phi(\eta + \xi_1) - \phi(\eta))}{G^{\omega}(\xi_1, \eta)} a_1(\xi_1, \eta),$$

$$\begin{aligned}\phi_m(\xi_1, \ldots, \xi_m, \eta) = \frac{1}{G^{\omega}(\xi_1, \ldots, \xi_m, \eta)} \Bigg(&(m+1)(\phi(\eta + \xi_1 + \cdots + \xi_m) \\ &- \phi(\eta)) a_m(\xi_1, \ldots, \xi_m, \eta) + \sum_{n=1}^{m-1} \Big\langle n\phi_n(\xi_1, \ldots, \xi_{n-1}, \xi_n + \cdots + \xi_m, \eta) a_{m-n} \\ &\times (\xi_n, \ldots, \xi_m) + (m - n + 1)\phi_n\left(\xi_1, \ldots, \xi_n, \eta + \sum_{l=n+1}^{m} \xi_l\right) a_{m-n}(\xi_{n+1}, \ldots, \xi_m, \eta) \\ &- (m - n + 1) a_{m-n}\left(\xi_{n+1}, \ldots, \xi_m, \eta + \sum_{l=1}^{n} \xi_l\right) \phi_n(\xi_1, \ldots, \xi_n, \eta) \Big\rangle \Bigg).\end{aligned}$$

The proof can be found in [6].

Theorem 2 and proposition 2 suggest the following integrability test for equation (4).

- Compute the symbolic representation of equation (4) and calculate the first few coefficients $\phi_i(\xi_1, \ldots, \xi_i, \eta)$, $i = 1, 2, \ldots$.
- Check the quasi-locality conditions.

In the following section we apply this test to isolate and classify integrable generalizations of the Camassa–Holm equation.

3. Lists of generalized Camassa–Holm-type equations

In this section, we present the classification results of Camassa–Holm-type equations with quadratic and cubic nonlinearity. We consider the following three ansätze for equation (4):

$$\begin{aligned}\left(1 - \epsilon^2 D_x^2\right) u_t = &c_1 u u_x + \epsilon \left[c_2 u u_{xx} + c_3 u_x^2\right] + \epsilon^2 \left[c_4 u u_{xxx} + c_5 u_x u_{xx}\right] \\ &+ \epsilon^3 \left[c_6 u u_{xxxx} + c_7 u_x u_{xxx} + c_8 u_{xx}^2\right] \\ &+ \epsilon^4 \left[c_9 u u_{xxxxx} + c_{10} u_x u_{xxxx} + c_{11} u_{xx} u_{xxx}\right],\end{aligned} \tag{9}$$

$$\begin{aligned}\left(1 - \epsilon^2 D_x^2\right) u_t = &c_1 u_x^2 + \epsilon c_2 u_x u_{xx} + \epsilon^2 \left[c_3 u_x u_{xxx} + c_4 u_{xx}^2\right] + \epsilon^3 \left[c_5 u_x u_{xxxx} + c_6 u_{xx} u_{xxx}\right] \\ &+ \epsilon^4 \left[c_7 u_x u_{xxxxx} + c_8 u_{xx} u_{xxxx} + c_9 u_{xxx}^2\right]\end{aligned} \tag{10}$$

and

$$\begin{aligned}\left(1 - \epsilon^2 D_x^2\right) u_t = &c_1 u^2 u_x + \epsilon \left[c_2 u^2 u_{xx} + c_3 u u_x^2\right] + \epsilon^2 \left[c_4 u^2 u_{xxx} + c_5 u u_x u_{xx} + c_6 u_x^3\right] \\ &+ \epsilon^3 \left[c_7 u^2 u_{xxxx} + c_8 u u_x u_{xxx} + c_9 u u_{xx}^2 + c_{10} u_x^2 u_{xx}\right] \\ &+ \epsilon^4 \left[c_{11} u^2 u_{xxxxx} + c_{12} u u_x u_{xxxx} + c_{13} u u_{xx} u_{xxx} + c_{14} u_x^2 u_{xxx} + c_{15} u_x u_{xx}^2\right].\end{aligned} \tag{11}$$

Here ϵ and c_i are the complex parameters and $\epsilon \neq 0$. The right-hand sides of equations (9), (10) and (11) are homogeneous differential polynomials of weights 1, 2 and 1, respectively, if we assume that weight of u_i is i, weight of ϵ equals -1 and weights of u_t in (9), (10) and (11) are 1, 2 and 1, respectively.

We bring equations (9)–(11) to the form (4) by shift transformation $u \to u + 1$ in the case of equations (9) and (11), and by $u \to u + x$ in the case of equation (10). We then construct the corresponding symbolic representations and compute first three coefficients of the corresponding formal recursion operators using proposition 2. In each class we then isolate the equations for which the first three coefficients of the corresponding formal recursion operators are quasi-local—this is the necessary integrability condition according to theorem 2. We then study the obtained equations in some details and present corresponding higher symmetries, Lax representations or linearization transformations.

3.1. Equations with quadratic nonlinearity

Theorem 3. *Suppose that at least one of the following equations is not satisfied:*

$$c_2 = 0, \qquad c_6 = 0, \qquad c_9 = 0, \qquad c_1 + c_4 = 0. \tag{12}$$

Then if equation (9) possesses an infinite hierarchy of quasi-local higher symmetries, then up to re-scaling $x \to \alpha x, t \to \beta t, u \to \gamma u, \alpha, \beta, \gamma = const$, it is one of the list:

$$\left(1 - \epsilon^2 D_x^2\right) u_t = 3uu_x - 2\epsilon^2 u_x u_{xx} - \epsilon^2 u u_{xxx}, \tag{13}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = D_x \left(4 - \epsilon^2 D_x^2\right) u^2, \tag{14}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = D_x \left[\left(4 - \epsilon^2 D_x^2\right) u\right]^2, \tag{15}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = D_x (2 + \epsilon D_x) \left[(2 - \epsilon D_x) u\right]^2, \tag{16}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = D_x (2 - \epsilon D_x)(1 + \epsilon D_x) u^2, \tag{17}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = D_x (2 - \epsilon D_x) \left[(1 + \epsilon D_x) u\right]^2, \tag{18}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = D_x \left[(2 - \epsilon D_x)(1 + \epsilon D_x) u\right]^2, \tag{19}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = D_x (1 + \epsilon D_x) \left[(2 - \epsilon D_x) u\right]^2, \tag{20}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = \left(1 - \epsilon^2 D_x^2\right)\left(\epsilon u u_{xx} - \tfrac{1}{2}\epsilon u_x^2 + c u u_x\right), \qquad c \in \mathbb{C}, \tag{21}$$

$$\left(1 - \epsilon^2 D_x^2\right) u_t = (1 - \epsilon D_x) \left[\epsilon S(u) S(u_{xx}) - \tfrac{1}{2}\epsilon (S(u_x))^2 - \tfrac{1}{2} c S(u) S(u_x)\right], \qquad S = 1 + \epsilon D_x. \tag{22}$$

We introduce a linear term into equation (9) by a shift $u \to u+1$ and construct the symbolic representation of the equation. The condition that at least one of the equations in (12) is not satisfied insures that $\omega(\xi_1) \neq \text{const}\ \xi_1$ in the corresponding symbolic representations. To prove the theorem it is sufficient to check the quasi-locality conditions of $\hat{u}\phi_1(\xi_1, \eta)$, $\hat{u}^2\phi_2(\xi_1, \xi_2, \eta)$ and $\hat{u}^3\phi_3(\xi_1, \xi_2, \xi_3, \eta)$ of the formal recursion operator:

$$\Lambda = \eta + \hat{u}\phi_1(\xi_1, \eta) + \hat{u}^2\phi_2(\xi_1, \xi_2, \eta) + \hat{u}^3\phi_3(\xi_1, \xi_2, \xi_3, \eta).$$

We do not present here the explicit formulae for these functions as they are quite cumbersome. One can easily compute them using proposition 2.

Theorem 4. *Suppose that at least one of the following equations is not satisfied:*

$$c_2 = 0, \qquad c_5 = 0, \qquad c_7 = 0, \qquad 2c_1 + c_3 = 0.$$

Then if equation (10) possesses an infinite hierarchy of quasi-local higher symmetries, then up to re-scaling $x \to \alpha x, t \to \beta t, u \to \gamma u, \alpha, \beta, \gamma = const$, it is one of the list:

$$\left(1 - \epsilon^2 D_x^2\right) u_t = \tfrac{1}{2}\left(3u_x^2 - 2\epsilon^2 u_x u_{xxx} - \epsilon^2 u_{xx}^2\right), \tag{23}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = \left(4-\epsilon^2 D_x^2\right) u_x^2, \tag{24}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = \left[\left(4-\epsilon^2 D_x^2\right)u_x\right]^2, \tag{25}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = (2+\epsilon D_x)\left[(2-\epsilon D_x)u_x\right]^2, \tag{26}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = (2-\epsilon D_x)(1+\epsilon D_x)u_x^2, \tag{27}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = (2-\epsilon D_x)\left[(1+\epsilon D_x)u_x\right]^2, \tag{28}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = \left[(2-\epsilon D_x)(1+\epsilon D_x)u_x\right]^2, \tag{29}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = (1+\epsilon D_x)\left[(2-\epsilon D_x)u_x\right]^2. \tag{30}$$

We introduce a linear term by a shift $u \to u + x$ and then construct the symbolic representation of equation (10). To prove the theorem it is again necessary to check the quasi-locality conditions of the first three terms of the corresponding formal recursion operator.

Let us consider now some properties of equations (13)–(22) and (23)–(30).

Camassa–Holm equation (13). Equation (13) is the Camassa–Holm equation. It can be rewritten as

$$m_t = 2mu_x + um_x, \qquad m = u - \epsilon^2 u_{xx}.$$

The Camassa–Holm equation possesses an infinite hierarchy of *local* higher symmetries and the first non-trivial local symmetry is

$$u_\tau = D_x(u - \epsilon^2 u_{xx})^{-\frac{1}{2}}.$$

The Lax representation and the bi-Hamiltonian structure can be found in [1, 4].

Degasperi–Procesi equation (14). Equation (14) is the Degasperis–Procesi equation and it can be rewritten as

$$m_t = 6mu_x + 2um_x, \qquad m = \left(1-\epsilon^2 D_x^2\right)u.$$

The Degasperis–Procesi equation also possesses an infinite hierarchy of local higher symmetries and the first such a non-trivial symmetry is

$$u_\tau = \left(4-\epsilon^2 D_x^2\right) D_x(u - \epsilon^2 u_{xx})^{-\frac{2}{3}}.$$

The bi-Hamiltonian structure and the Lax representation for the Degasperis–Procesi equation can be found in [5].

Equation (15). The first non-trivial symmetry of equation (15) is

$$u_\tau = D_x\left[\left(4-\epsilon^2 D_x^2\right)\left(1-\epsilon^2 D_x^2\right)u\right]^{-\frac{2}{3}}.$$

Equation (15) can be rewritten as

$$m_t = D_x\,(m+3u)^2, \qquad m = u - \epsilon^2 u_{xx}.$$

It is easy to see that the Degasperis–Procesi equation transforms into equation (15) under the transformation

$$u \to \left(4-\epsilon^2 D_x^2\right)u.$$

The Lax representation for equation (15) is

$$\psi_x - \psi_{xxx} - \lambda(4m - \epsilon^2 m_{xx})\psi = 0,$$

$$\psi_t = \frac{2}{\lambda}\psi_{xx} + 2(m+3u)\psi_x - 2\left(m_x + 3u_x + \frac{2}{3\lambda}\right)\psi.$$

Equation (16). The first non-trivial symmetry of equation (16) is

$$u_\tau = (2+\epsilon D_x)D_x[(2-\epsilon D_x)(u-\epsilon^2 u_{xx})]^{-\frac{2}{3}}.$$

The Degasperis–Procesi equation transforms into (16) under the change of variables

$$u \to (2-\epsilon D_x)u.$$

The Lax representation for equation (16) is

$$\psi_x - \psi_{xxx} - \lambda\,(2m-\epsilon m_x)\,\psi = 0, \qquad m = u - \epsilon^2 u_{xx},$$
$$\psi_t = \frac{2}{\lambda}\psi_{xx} + 2(2u-\epsilon u_x)\psi_x - 2\left(2u_x - \epsilon u_{xx} + \frac{2}{3\lambda}\right)\psi.$$

Note that the other transformation $u \to (2+\epsilon D_x)u$ of Degasperis–Procesi gives the equation $\left(1-\epsilon^2 D_x^2\right)u_t = D_x(2-\epsilon D_x)\,[(2+\epsilon D_x)u]^2$, which transforms into (16) under the change $x \to -x, t \to -t$.

Equation (17). Equation (17) possesses a hierarchy of local higher symmetries and the first non-trivial one is

$$u_\tau = D_x\,[(1-\epsilon D_x)u]^{-1}.$$

The last equation is linearizable by the transformation

$$x = -\epsilon\log(v_y(y,t)), \qquad u = \frac{1}{\sqrt{\epsilon}\log(v(y,t))_y}, \quad \Longrightarrow \quad v_t = v_{yy}.$$

Equation (18). The higher symmetries of this equation are quasi-local and the first non-trivial one is

$$(1+\epsilon D_x)u_\tau = D_x\left[\left(1-\epsilon^2 D_x^2\right)u\right]^{-1}.$$

However, equation (18) can be rewritten as

$$m_t = D_x(2-\epsilon D_x)\,[(1+\epsilon D_x)u]^2, \qquad m = u - \epsilon^2 u_{xx}$$

and the latter equation possesses an infinite hierarchy of local higher symmetries in dynamical variable m. One can easily check that the first such a symmetry is

$$m_\tau = D_x(1-\epsilon D_x)m^{-1}.$$

The last equation is linearizable by the transformation

$$x = -\epsilon\log(v(y,t)), \qquad m = -\frac{1}{\sqrt{\epsilon}\log(v(y,t))_y}, \quad \Longrightarrow \quad v_t = v_{yy}.$$

Equations (17) and (18) are related by the transformation $u \to (1+\epsilon D_x)u$. It is clear that this transformation does not preserve the locality of higher symmetries of equation (17).

Equation (19). The first non-trivial higher symmetry of this equation is quasi-local

$$(1+\epsilon D_x)u_\tau = D_x[(2-\epsilon D_x)(u-\epsilon^2 u_{xx})]^{-2}.$$

However equation (19) can be written as

$$m_t = D_x[(2-\epsilon D_x)(1+\epsilon D_x)u]^2, \qquad m = u - \epsilon^2 u_{xx}$$

and the latter equation possesses an infinite hierarchy of local higher symmetries and the first one reads $m_\tau = D_x(1-\epsilon D_x)\,[(2-\epsilon D_x)m]^{-2}$. The Lax representation for equation (19) is not known yet.

Equation (20). The first non-trivial higher symmetry of equation (20) is

$$u_\tau = D_x\,[(2-\epsilon D_x)(1-\epsilon D_x)u]^{-2}.$$

This equation possesses an infinite hierarchy of local higher symmetries. Note that equation (19) can be obtained from (20) by the transformation $u \to (1+\epsilon D_x)u$. The Lax representation for this equation is not known yet.

Equation (21) is a local second-order linearizable evolutionary equation [10], while equation (22) transforms into (21) as $u \to (1+\epsilon D_x)u$.

Equations (23)–(30) can be obtained from equations (13)–(20) via the potentiation transformation $u \to u_x$. Indeed, the right-hand side of each of equations (13)–(20) is a total x-derivative, and therefore these equations admit a non-invertible transformation $u = \hat{u}_x$. For example, in the case of the Camassa–Holm equation (13) we have

$$\left(1-\epsilon^2 D_x^2\right)u_t = 3uu_x - 2\epsilon^2 u_x u_{xx} - \epsilon^2 u u_{xxx} = D_x\left(\tfrac{3}{2}u^2 - \epsilon^2 u u_{xx} - \tfrac{1}{2}\epsilon^2 u_x^2\right),$$

and therefore if $u = \hat{u}_x$, then for $\hat{u}$ we obtain equation (23):

$$\left(1-\epsilon^2 D_x^2\right)\hat{u}_t = \left(\tfrac{3}{2}\hat{u}_x^2 - \epsilon^2 \hat{u}_x \hat{u}_{xxx} - \tfrac{1}{2}\epsilon^2 \hat{u}_{xx}^2\right).$$

3.2. Equations with cubic nonlinearity

Now we consider equations with cubic nonlinearity.

Theorem 5. *Suppose that at least one of the following equations is not satisfied:*

$$c_2 = 0, \qquad c_7 = 0, \qquad c_{11} = 0, \qquad c_1 + c_4 = 0.$$

Then if equation (11) possesses an infinite hierarchy of quasi-local higher symmetries, then up to re-scaling $x \to \alpha x, t \to \beta t, u \to \gamma u, \alpha, \beta, \gamma = const$, it is one of the list:

$$\left(1-\epsilon^2 D_x^2\right)u_t = \epsilon^2 u^2 u_{xxx} + 3\epsilon^2 u u_x u_{xx} - 4u^2 u_x, \tag{31}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = D_x\left(\epsilon^2 u^2 u_{xx} - \epsilon^4 u_x^2 u_{xx} + \epsilon^2 u u_x^2 - u^3\right), \tag{32}$$

$$\begin{aligned}\left(1-\epsilon^2 D_x^2\right)u_t = {}& \epsilon^4 u_x^2 u_{xxx} + \epsilon^4 u_x u_{xx}^2 + 2\epsilon^3 u u_x u_{xxx} + \epsilon^3 u u_{xx}^2 + \epsilon^3 u_x^2 u_{xx} \\ &+ \epsilon^2 u^2 u_{xxx} - \epsilon^2 u_x^3 - \epsilon u^2 u_{xx} - 3\epsilon u u_x^2 - 2u^2 u_x,\end{aligned} \tag{33}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = (1+\epsilon D_x)\left(\epsilon u^2 u_{xx} + \epsilon u u_x^2 - 2u^2 u_x\right), \tag{34}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = (1+\epsilon D_x)\left(2\epsilon^3 u_x^2 u_{xx} - \epsilon^2 u u_x u_{xx} - \epsilon^2 u_x^3 - \epsilon u^2 u_{xx} - \epsilon u u_x^2 + 2u^2 u_x\right), \tag{35}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = \left(1-\epsilon^2 D_x^2\right)\left(\epsilon^2 u^2 u_{xxx} - \epsilon^2 u u_x u_{xx} + \tfrac{4}{9}\epsilon^2 u_x^3 + c u^2 u_x\right), \qquad c \in \mathbb{C}, \tag{36}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = \left(1-\epsilon^2 D_x^2\right)\left(\epsilon^2 u^2 u_{xxx} + \epsilon^2 u u_x u_{xx} - \tfrac{2}{9}\epsilon^2 u_x^3 + c u^2 u_x\right), \qquad c \in \mathbb{C}, \tag{37}$$

$$\begin{aligned}\left(1-\epsilon^2 D_x^2\right)u_t = {}& \left(1-\epsilon^2 D_x^2\right)\left(\epsilon^2 u^2 u_{xxx} + \epsilon^2 u u_x u_{xx} - \tfrac{2}{9}\epsilon^2 u_x^3\right. \\ &\left.+ 3c\epsilon u^2 u_{xx} + c\epsilon u u_x^2 + 2c^2 u^2 u_x\right), \qquad c \in \mathbb{C},\end{aligned} \tag{38}$$

$$\begin{aligned}\left(1-\epsilon^2 D_x^2\right)u_t = {}& \left(1-\epsilon^2 D_x^2\right)\left(\epsilon^2 u^2 u_{xxx} + \tfrac{1}{9}\epsilon^2 u_x^3 + 3c\epsilon u^2 u_{xx} + c\epsilon u u_x^2 + 2c^2 u^2 u_x\right), \\ & c \in \mathbb{C},\end{aligned} \tag{39}$$

$$\left(1-\epsilon^2 D_x^2\right)u_t = \left(1-\epsilon^2 D_x^2\right)\left(\epsilon u^2 u_{xx} + c u^2 u_x\right), \qquad c \in \mathbb{C}. \tag{40}$$

We introduce a linear term into equation (11) by a shift transformation $utou+1$. The proof requires to check the quasi-locality conditions of the first three terms of the formal recursion operator.

Equation (31). The first local higher symmetry of this equation is

$$u_\tau = m^{-\frac{13}{3}}\epsilon^2\,(mm_{xxx} - 5m_x m_{xx}) + \tfrac{40}{9}\epsilon^2 m^{-\frac{16}{3}}m_x^3 - 4m^{-\frac{10}{3}}m_x, \qquad m = u - \epsilon^2 u_{xx}.$$

Equation (31) can be rewritten as

$$m_t = -(u^2 m_x + 3muu_x), \qquad m = u - \epsilon^2 u_{xx}.$$

The Lax representation for equation (31) is

$$\epsilon^3\psi_{xxx} = \epsilon\psi_x + \lambda m^2\psi + 2\epsilon^3\frac{m_x}{m}\psi_{xx} + \frac{mm_{xx} - 2m_x^2}{m^2}\psi_x,$$

$$\psi_t = \frac{\epsilon}{\lambda}\frac{u}{m}\psi_{xx} - \frac{\epsilon}{\lambda}\frac{mu_x + um_x}{m^2}\psi_x - u^2\psi_x.$$

Equation (31) has been recently studied in detail in [7], where the Lax representation was constructed in a different form. The authors of [7, 11] also obtained the bi-Hamiltonian structure and constructed the peakon solutions for equation (31), for which the positions and amplitudes of the peaks satisfy a finite-dimensional integrable Hamiltonian system.

Equation (32). The first local higher symmetry of equation (32) is

$$u_\tau = m^{-3}m_x, \qquad m = u - \epsilon^2 u_{xx}.$$

Equation (32) can be rewritten as

$$m_t = \left(\epsilon^2 u_x^2 - u^2\right)m_x - 2m^2 u_x.$$

This equation was recently derived from shallow water theory in [8], where the Lax representation and bi-Hamiltonian structure were presented and different types of solutions were constructed; however, an equivalent form of this equation was given by Fokas in [13].

Equation (33). The higher symmetries of this equation are quasi-local and the first one reads

$$(1 + \epsilon D_x)u_\tau = m^{-7}\left(\epsilon mm_{xx} - 3\epsilon m_x^2 - 2mm_x\right), \qquad m = u - \epsilon^2 u_{xx}.$$

Equation (33) can be rewritten as

$$m_t = -\epsilon^2 u_x^2 m_x - 2muu_x + m^2 u_x - 2\epsilon uu_x m_x + \frac{1}{\epsilon}mu(m-u) - \epsilon mu_x^2 - u^2 m_x$$

and the latter equation possesses an infinite hierarchy of local higher symmetries in m. The first such symmetry is

$$m_\tau = (1 - \epsilon D_x)m^{-7}\left(\epsilon mm_{xx} - 3\epsilon m_x^2 - 2mm_x\right), \qquad m = u - \epsilon^2 u_{xx}.$$

Equation (34). Equation (34) possesses an infinite hierarchy of local higher symmetries and the first non-trivial one is

$$u_\tau = v^{-7}\left(\epsilon vv_{xx} - 3\epsilon v_x^2 - 2vv_x\right), \qquad v = u - \epsilon u_x.$$

Equation (35). The first local higher symmetry of this equation is

$$u_\tau = v^{-2}(v + \epsilon v_x)^{-1} - v^{-3}, \qquad v = u - \epsilon u_x.$$

The latter equation is linearizable as it is a second-order integrable evolution equation (cf equations (17) and (18)).

Equations (37)–(40) correspond to local evolutionary equations of orders 3 and 2.

4. Conclusions

In this article, we have considered polynomial homogeneous generalizations of the Camassa–Holm-type equation with quadratic and cubic nonlinearity. We have classified all equations of the form (9), (10) and (11), which possess infinite hierarchies of (quasi)-local higher symmetries. We have shown that the obtained equations can be treated as non-local symmetries of local scalar evolution quasi-linear integrable equations of orders 2, 3 and 5.

Some of the obtained equations seem to be new and are likely to provide more examples of solution phenomena (peakons, compactons, other weak/non-classical solutions) that do not appear in local evolution equations [14]. The study of multi-phase solutions of these equations remains out of the scope of this paper.

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